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LETTER TO THE EDITOR

Variational principle for the turbulent diffusion equation

Andrzej Icha

Institute of Oceanology, Polish Academy of Sciences 81-967 Sopot, PO Box 68, ul Powstańców Warszawy 55, Poland

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Abstract. The application of functional formalism to the description of turbulent diffusion is shown. The problem of the turbulent diffusion is formulated on the basis of a variational principle for the Φ equation in the appropriate functional space. The existence of an exact constant of motion, as a consequence of this principle, is shown.

A functional formalism for turbulent diffusion which evolves in time according to the passive scalar transport equation has been given by Szafirski (1971), who introduced the characteristic functional Φ of a probability distribution of the velocity and concentration fields and derived a functional differential equation for Φ.

Finding the solution of the initial problem for the Φ equation involves great mathematical difficulties. For several years only solutions simplifying the Hopf-type equations were found (cf Alankus 1988, Icha 1989a, b, Szafirski 1970, 1971). Thus, it is important to show some new methods of solving the Φ equation.

In this letter we show how a variational technique may be used to study the problem of turbulent diffusion. We derive a Φ equation for the space characteristic functional using the appropriate variational principle in the suitable functional space and we show that a consequence of this principle is the existence of an exact constant of the motion.

The equation that governs the evolution of a concentration $c(x, t)$ subject to diffusion and convection by an incompressible fluid is (Landau and Lifshitz 1986)

$$\frac{\partial c}{\partial t} = -\frac{\partial}{\partial x_i}(u_i c) + k\frac{\partial^2 c}{\partial x_i\, \partial x_i}. \tag{1}$$

Here $x \in \mathbb{R}^3$, $t \in [0; \infty)$, $u = u(x, t)$ denotes the solenoidal flow velocity and k is the molecular diffusivity.

The full statistical description of the field $[c, u]$ can be obtained by defining the space characteristic functional Φ (Szafirski 1971)

$$\begin{aligned}\Phi[s, p; t] &= \left\langle \exp\left[\mathrm{i}\int s(x)c(x, t)\,\mathrm{d}x + \mathrm{i}\int p_k(x, t)u_k(x, t)\,\mathrm{d}x\,\mathrm{d}t\right]\right\rangle \\ &= \int \exp\left[\mathrm{i}\int s(x)c(x, t)\,\mathrm{d}x + \mathrm{i}\int p_k(x, t)u_k(x, t)\,\mathrm{d}x\,\mathrm{d}t\right]\mathrm{d}P \end{aligned} \tag{2}$$

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where $s(x)$ is a continuous function with compact support in $D \subset \mathbb{R}^3$, $p(x, t)$ is a continuous vector with compact support in $D \times (0: \infty)$ and P denotes a probability measure of the random field $[c, u]$. This functional has the following properties:

$$\Phi[0, 0; t] = 1 : \Phi^*[s, p; t] = \Phi[-s, -p; t] : |\Phi[s, p; t]| \leq 1$$
$$\Phi[s, p; t] = \Phi[s, p^s; t] \tag{3}$$

where * denotes the complex conjugate and p^s is a solenoidal part of the real vector field $p(x, t)$.

The functional Φ satisfies the following first-order time-linear functional equation (Szafirski 1971):

$$\frac{\partial \Phi}{\partial t} = \int s(x)\left(\mathrm{i}\frac{\partial}{\partial x_k}\frac{\delta^2\Phi}{\delta p_k\,\delta s} + k\frac{\partial^2}{\partial x_k\,\partial x_k}\frac{\delta\Phi}{\delta s}\right)\mathrm{d}x \tag{4}$$

where $\delta/\delta(\cdot)$ is an operator of the functional derivative.

Let us now introduce the real functional differential operators

$$C = k\int s(x)\frac{\partial^2}{\partial x_k\,\partial x_k}\frac{\delta}{\delta s}\,\mathrm{d}x \qquad D = \int s(x)\frac{\partial}{\partial x_k}\frac{\delta^2}{\delta p_k\,\delta s}\,\mathrm{d}x. \tag{5}$$

Substituting (5) into (4), we obtain the equation for Φ in the form

$$\frac{\partial \Phi}{\partial t} = (C + \mathrm{i}D)\Phi. \tag{4'}$$

Later on we introduce the notion of the functional integral (Rzewuski 1969). As is well known, the functionals may be considered as the functions of an infinite number of variables (Rzewuski 1969, Volterra 1959). Given a functional $\Phi[s, p]$ we consider those functions $s(x)$, $p(x)$ which may be expanded in an orthogonal series

$$s(x) = \sum_n s_n\psi_n(x) \qquad p(x) = \sum_n p_n\Lambda_n(x) \tag{6}$$

where the set $\{\psi_n(x), \Lambda_n(x)\}$ is an infinite set of functions satisfying the orthonormality relations

$$\int \psi_n(x)\psi_m(x)\,\mathrm{d}x \equiv \delta_{nm} \qquad \int \Lambda_n(x)\Lambda_m(x)\,\mathrm{d}x \equiv \delta_{nm} \tag{7}$$

and the completeness relations

$$\sum_n \psi_n(x)\psi_n(y)\,\mathrm{d}x \equiv \delta(x-y) \qquad \sum_n \Lambda_n(x)\Lambda_n(y)\,\mathrm{d}x \equiv \delta(x-y). \tag{8}$$

Here δ_{nm}, $\delta(\cdot)$ are the Kronecker delta and the Dirac delta function, respectively.

The functions (6) form a subset of the set of all functions. For a given set of orthonormal functions $\psi_n(x)$, $\Lambda_n(x)$ they are determined uniquely by the infinite sets of expansion coefficients s_n, p_n. Introducing (6) into the functional $\Phi[s, p]$, we obtain a function of the infinite set $\{s_n, p_n\}$:

$$\Phi[s, p] \equiv \Phi\left[\sum_n s_n\psi_n, \sum_n p_n\Lambda_n\right] \equiv F(s_1, s_2, \ldots, s_i, \ldots, p_1, p_2, \ldots, p_i, \ldots). \tag{9}$$

According to the representation (9) of functionals by means of the function of infinitely many variables we can write the definition of a functional integral in the following way (Rzewuski 1969):

$$\int \Phi[s, p]D(s, p) \overset{\mathrm{def}}{\equiv} \prod_{i=1}^{\infty}\int \mathrm{d}s_i\,\mathrm{d}p_i F(s_1, s_2, \ldots, p_1, p_2, \ldots) \tag{10}$$

with the assumption that the integral on the right-hand side of (10) exists. This integral may be considered as the limit of an n-fold integral

$$\prod_{i=1}^{\infty} \int \mathrm{d}s_i\, \mathrm{d}p_i\, F(s_1, s_2, \ldots, p_1, p_2, \ldots)$$
$$= \lim_{n\to\infty} \prod_{i=1}^{n} \int \mathrm{d}s_i\, \mathrm{d}p_i\, F(s_1, \ldots, s_n, p_1, \ldots, p_n) \tag{11}$$

where

$$F(s_1, \ldots, s_n, p_1, \ldots, p_n) = \Phi\left[\sum_{i=1}^{n} s_i\psi_{i'} \sum_{i=1}^{n} p_i\Lambda_i\right]. \tag{12}$$

If the limit (11) exists, and is independent of the choice of the orthogonal sets $\{\psi_i, \Lambda_i\}$, we have in (10) a unique definition of a functional integral (Rzewuski 1969).

The representation (9) is valid only for functions which belong to Hilbert space. As the functions s, p are continuous functions with compact support, thus they belong to $L^2(\mathbb{R}^3)$ i.e. $\|\bar{s}\|^2 = \int \bar{s}\bar{s}\, \mathrm{d}x < \infty$, where $\bar{s} = (s, p)$, and the set $\{\psi_i, \Lambda_i\}$ satisfy the convergence condition $\Sigma\, \bar{\psi}_i^2 < \infty$, where $\bar{\psi}_i = \{\psi_i, \Lambda_i\}$.

Let us introduce the inner product in the functional space of the functionals $\Phi[s, p]$

$$(\Phi_1, \Phi_2) = \oint \Phi_1^* \Phi_2\, \mathrm{D}(s, p) \tag{13}$$

where the integral on the right-hand side of (13) is taken in the sense of the definition (10). As the inner product is finite, we find that

$$\begin{aligned} &\int \left(\frac{\delta\Phi_1^*}{\delta p_i}\Phi_2 + \Phi_1^* \frac{\delta\Phi_2}{\delta p_i}\right) \mathrm{D}(s, p) = 0 \\ &\int \left(\frac{\delta\Phi_1^*}{\delta s}\Phi_2 + \Phi_1^* \frac{\delta\Phi_2}{\delta s}\right) \mathrm{D}(s, p) = 0. \end{aligned} \tag{14}$$

We introduce the definition of the adjoints of operators (5) as follows:

$$(\Phi_1, C\Phi_2) = (C^\dagger\Phi_1, \Phi_2) \qquad (\Phi_1, D\Phi_2) = (D^\dagger\Phi_1, \Phi_2). \tag{15}$$

Taking into account (14) we obtain

$$C^\dagger = -C + a \qquad D^\dagger = D \tag{16}$$

where $a = -3k \int \Delta\delta(x)|_{x=0}\, \mathrm{d}x$ is a positive real constant (equal to $3kK^5/10\pi^2$, where K is the wavenumber).

We shall try to formulate the variational principle for equation (4′). Consider the action J in the time interval (t_0, t), of the form

$$J[\Phi] = \int_{t_0}^{t} \mathrm{e}^{-at}\, (\Phi, \dot{\Phi} - C\Phi - \mathrm{i}D\Phi)\, \mathrm{d}t \tag{17}$$

where the dot represents the derivative with respect to time t.

In expression (17), we may call $L(t, \Phi, \dot{\Phi}) = \mathrm{e}^{-at}(\Phi, \dot{\Phi} - C\Phi - \mathrm{i}D\Phi)$ the density of the Lagrangian. The equations of motion are the Euler equations obtained by variation of L with respect to the variables Φ and $\dot{\Phi}$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\Phi}}\right) - \frac{\partial L}{\partial \Phi} = 0. \tag{18}$$

In the particular case when $L = \mathrm{e}^{-at}(\Phi, \dot{\Phi} - C\Phi - \mathrm{i}D\Phi)$, we have

$$\mathrm{e}^{-at}(\Phi, \dot{\Phi} - C\Phi - \mathrm{i}D\Phi) = \mathrm{e}^{-at}(\Phi, \dot{\Phi}) + \mathrm{e}^{-at}(-C^{\dagger}\Phi, \Phi) + \mathrm{e}^{-at}(-\mathrm{i}D^{\dagger}\Phi, \Phi)$$

$$\frac{\partial L}{\partial \dot{\Phi}} = \mathrm{e}^{-at} \int \Phi^* \, \mathrm{D}(s, p)$$

$$\frac{\partial L}{\partial \Phi} = -\mathrm{e}^{-at} \int (C^{\dagger}\Phi)^* \, \mathrm{D}(s, p) - \mathrm{i}\, \mathrm{e}^{-at} \int (D^{\dagger}\Phi)^* \mathrm{D}(s, p)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\Phi}}\right) = -a\, \mathrm{e}^{-at} \int \Phi^* \mathrm{D}(s, p) + \mathrm{e}^{-at} \int \dot{\Phi}^* \mathrm{D}(s, p).$$

Substituting these expressions into (18), we obtain

$$-\mathrm{e}^{-at} \int (-\dot{\Phi}^* + a\Phi^* - C^{\dagger}\Phi^* - \mathrm{i}D^{\dagger}\Phi^*)\mathrm{D}(s, p) = 0.$$

Therefore,

$$-\dot{\Phi}^* + a\Phi^* - C^{\dagger}\Phi^* - \mathrm{i}D^{\dagger}\Phi^* = 0 \qquad -\dot{\Phi}^* + C\Phi^* - \mathrm{i}D\Phi^* = 0. \tag{19}$$

Comparing this equation with equation (4′) it can be seen that (19) is a complex conjugate of (4′). Hence equation (19) is equivalent to (4′).

Consider now the quantity

$$(\Phi, \Phi) = \int |\Phi[s, p; t]|^2 \mathrm{D}(s, p) \tag{20}$$

and examine the time evolution of (Φ, Φ) in time. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\Phi, \Phi) &= \frac{\mathrm{d}}{\mathrm{d}t} \int \Phi^*\Phi \; \mathrm{D(s, p)} = \int \dot{\Phi}^*\Phi \; \mathrm{D}(s, p) + \int \Phi^*\dot{\Phi} \; \mathrm{D}(s, p) \\
&= \int [(C\Phi^*)\Phi - \mathrm{i}(D\Phi^*)\Phi]\mathrm{D}(s, p) + \int [\Phi^* C\Phi + \mathrm{i}\, \Phi^* D\Phi]\mathrm{D}(s, p) \\
&= \int [(C\Phi)^*\Phi - \mathrm{i}(D\Phi)^*\Phi]\mathrm{D}(s, p) + \int [\Phi^* C\Phi + \mathrm{i}\Phi^* D\Phi]\mathrm{D}(s, p) \\
&= (C\Phi, \Phi) - \mathrm{i}(D\Phi, \Phi) + (\Phi, C\Phi) + \mathrm{i}(\Phi, D\Phi) \\
&= (C\Phi, \Phi) - (C\Phi, \Phi) + a(\Phi, \Phi) + \mathrm{i}(\Phi, D\Phi) - \mathrm{i}(\Phi, D\Phi) = a(\Phi, \Phi).
\end{aligned} \tag{21}$$

We show that the equation (4′) has a constant of motion

$$\Pi = \frac{(\Phi, \dot{\Phi})}{(\Phi, \Phi)} \tag{22}$$

i.e. $\mathrm{d}\Pi/\mathrm{d}t = 0$.

Indeed, we have

$$\begin{aligned}
\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(\Phi, \dot{\Phi})}{(\Phi, \Phi)}\right) &= \frac{[(\dot{\Phi}, \dot{\Phi}) + (\Phi, \ddot{\Phi})] * (\Phi, \Phi) - 2(\dot{\Phi}, \Phi)(\dot{\Phi}, \Phi)}{(\Phi, \Phi)^2} \\
&= \frac{[(\dot{\Phi}, \dot{\Phi}) + (\Phi, \ddot{\Phi})] * (\Phi, \Phi) - (\mathrm{d}/\mathrm{d}t)[(\Phi, \dot{\Phi}) * (\Phi, \dot{\Phi})]}{(\Phi, \Phi)^2}.
\end{aligned}$$

But

$$(\Phi, \dot{\Phi}) = (\Phi, C\dot{\Phi} + \mathrm{i}D\dot{\Phi}) = (\Phi, C\dot{\Phi}) + (\Phi, \mathrm{i}D\dot{\Phi}) = (C^{\dagger}\Phi, \dot{\Phi}) + \mathrm{i}(D^{\dagger}\Phi, \dot{\Phi})$$
$$= -(C\Phi, \dot{\Phi}) + a(\Phi, \dot{\Phi}) + \mathrm{i}(D\Phi, \dot{\Phi}) = a(\Phi, \dot{\Phi}) - [(C\Phi, \dot{\Phi}) - \mathrm{i}(D\Phi, \dot{\Phi})]$$
$$= a(\Phi, \dot{\Phi}) - (C\Phi + \mathrm{i}D\Phi, \dot{\Phi} = a(\Phi, \dot{\Phi}) - (\dot{\Phi}, \dot{\Phi})$$

(because $(\alpha\Phi_1, \Phi_2) = \alpha^*(\Phi_1, \Phi_2)$).

Taking into account this fact and the relation (21) we obtain finally

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \frac{a(\Phi, \dot{\Phi}) * (\Phi, \Phi) - a(\Phi, \dot{\Phi}) * (\Phi, \Phi)}{(\Phi, \Phi)^2} = 0. \tag{23}$$

The results presented in this letter are preliminary. In particular it remains to consider the physical interpretation of the constancy of the quantity Π and an evaluation of the value of Π for the well known characteristic functionals of Poisson and Gaussian fields. These problems shall be presented in future papers.

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